Failure theories

Lecture 6 – failure theories

Introduction

- Uniaxial tensile test provides information about the constitutive response of a material under simple stress state.
- Real components are subjected to complex stress state: multiaxial state of stress
- The possibility to experimentally probe the material under multi-axial state of stress is limited.
- We need to follow a different approach: to find, if exists, the way to calculate an “equivalent” uniaxial stress that causes the same effect as for the given multi-axial state of stress

Introduction

- From the mathematical point of view, this means finding a relationship such as:

$$\sigma_{eq} = f(\sigma_{ij})$$

- To determine such function, the following approach is followed:
  1. Make a hypothesis of equivalence
  2. Derive the expression of σ_{eq} for the generic three dimensional stress state
  3. Derive the expression of σ_{eq} for a biaxial (plane) stress state as a function of in plane components (σ_x, σ_y, τ_{xy})
  4. Calculate the ratio between the limit states σ_L/τ_L

- Limit state indicates the maximum stress state that the material can tolerate before failure
- These values also indicated as "allowables"
- Depends on the material behavior
- For static loads,
  - Ductile materials:

$$\sigma_L = \sigma_Y$$

$$\tau_L = \tau_Y$$

  - Brittle materials

$$\sigma_L = \sigma_R$$

$$\tau_L = \tau_R$$

Failure theories

- These relationships or criteria are also known as failure theories since they provide the equivalence relationship between two “critical” stress states: the uniaxial and the multiaxial.

- Different equivalence relationship have been proposed based on material response observed in experiments

- Non of them is better than the other: some fit better than other for specific material classes!

Failure theories: Maximum normal stress (or Rankine criterion)

1. Assumption: the limit state is predicted to occur at the material point when the maximum principal stress reaches the limit value σ_L

2. For the generic multiaxial state of stress the critical condition becomes:

$$\sigma_1 = \sigma_L \qquad \text{tensile}$$

$$\sigma_3 = \sigma_L \qquad \text{compression}$$

Similar condition is obtained for the uniaxial stress state (only one stress component)

$$\sigma_{eq} = \sigma_1 \qquad \text{tensile}$$

$$\sigma_{eq} = \sigma_3 \qquad \text{compression}$$

Therefore, the limit state (failure) is predicted to occur when:

$$\sigma_{eq} \geq \sigma_L$$

3. In the case of plane stress, the equivalent stress as a function of the in plane stress components is obtained from the Mohr circle:

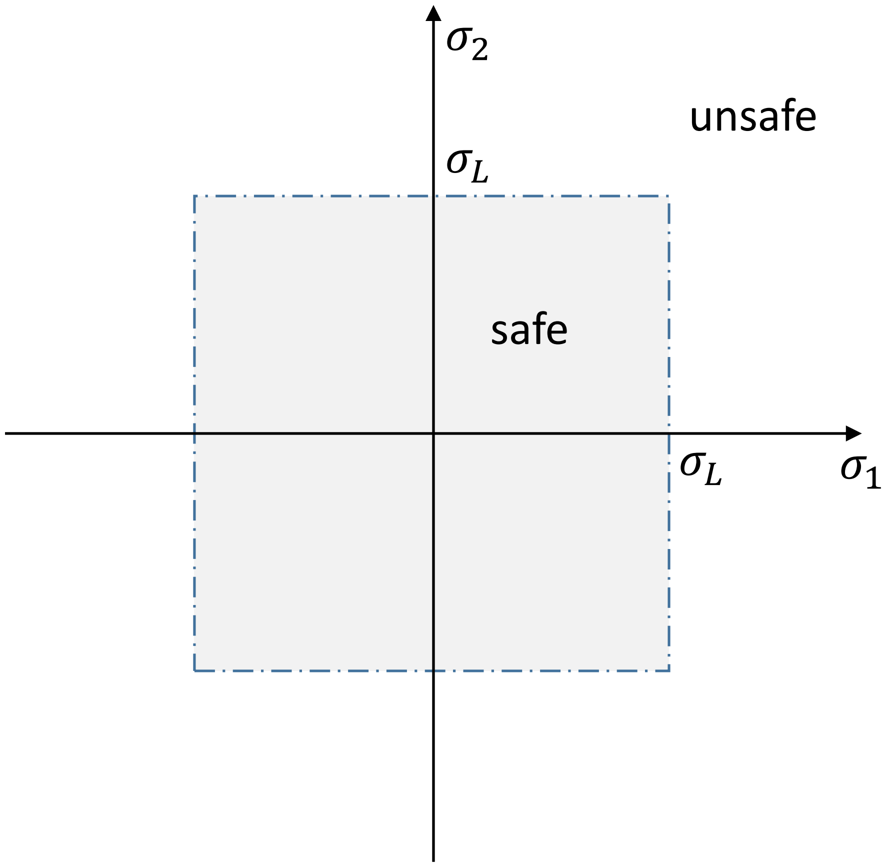
$$\sigma_{eq} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

4. For torsion or simple shear:

$$\sigma_{eq} = \tau_{xy}$$

At failure: $\sigma_{eq} = \tau_L = \sigma_L \rightarrow \dfrac{\sigma_L}{\tau_L} = 1$

Failure theories: Maximum normal stress (or Rankine criterion)



Failure theories: Maximum deformation (or Saint-venant criterion)

1. Assumption: the limit state is predicted to occur at the material point when the maximum principal deformation reaches the limit value ε_L

2. For the generic multiaxial state of stress the critical condition becomes:

$$\varepsilon_1 = \frac{1}{E}[\sigma_1 - \nu(\sigma_2 + \sigma_3)] = \varepsilon_L \quad \text{tensile}$$

$$\varepsilon_3 = \frac{1}{E}[\sigma_3 - \nu(\sigma_2 + \sigma_1)] = \varepsilon_L \quad \text{compression}$$

For the uniaxial case:

$$\varepsilon_1 = \sigma_{eq}/E \rightarrow \varepsilon_L = \sigma_L/E$$

Therefore, the limit state (failure) is predicted to occur when:

$$\sigma_{eq} \geq \sigma_L$$

3. In the case of plane stress:

$$\sigma_{eq} = (1-\nu)\frac{\sigma_x + \sigma_y}{2} + (1+\nu)\sqrt{\left(\frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

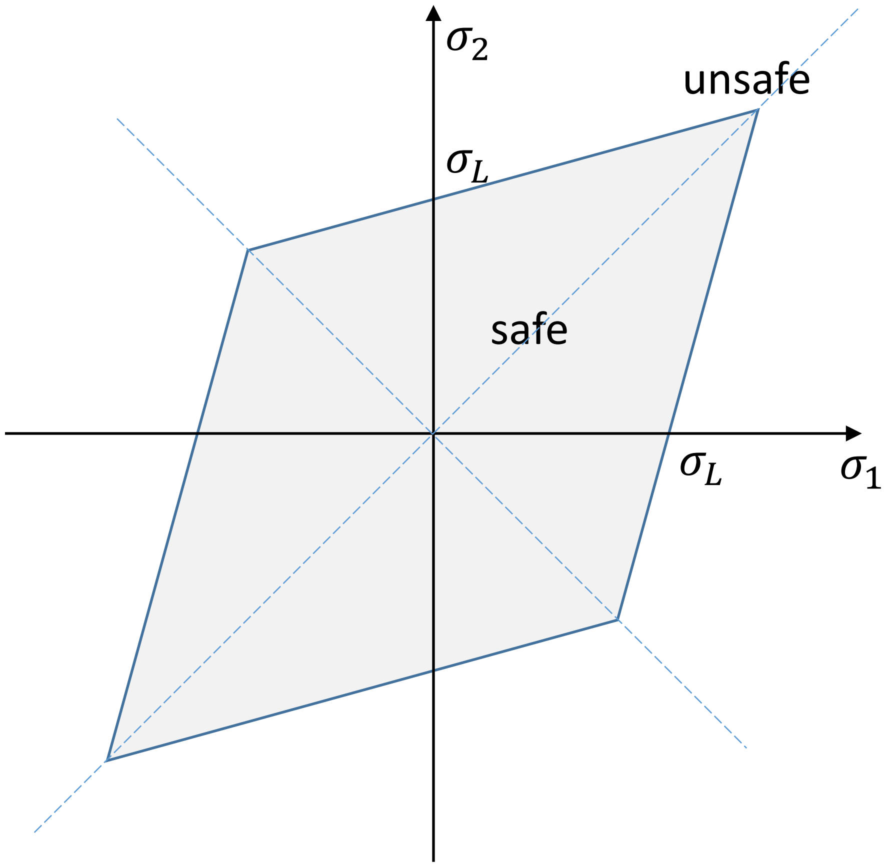
4. For torsion or simple shear:

$$\sigma_{eq} = (1+\nu)\tau_{xy}$$

At failure:

$$\sigma_{eq} = (1+\nu)\tau_L = \sigma_L \rightarrow \frac{\sigma_L}{\tau_L} = (1+\nu)$$

Failure theories: Maximum deformation (or Saint-venant criterion)



Failure theories: Maximum shear (or Tresca criterion)

1. Assumption: the limit state is predicted to occur at the material point when the maximum shear stress reaches the limit value τ_L

2. For the generic multiaxial state of stress the critical condition becomes:

$$\tau_{max} = \frac{1}{2}[\sigma_1 - \sigma_3] = \tau_L$$

For the uniaxial case:

$$\tau_{max} = \sigma_{eq}/2 \rightarrow \tau_L = \sigma_L/2$$

Therefore, the limit state (failure) is predicted to occur when:

$$\sigma_{eq} \geq \sigma_L$$

3. In the case of plane stress:

$$\sigma_{eq} = \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$$

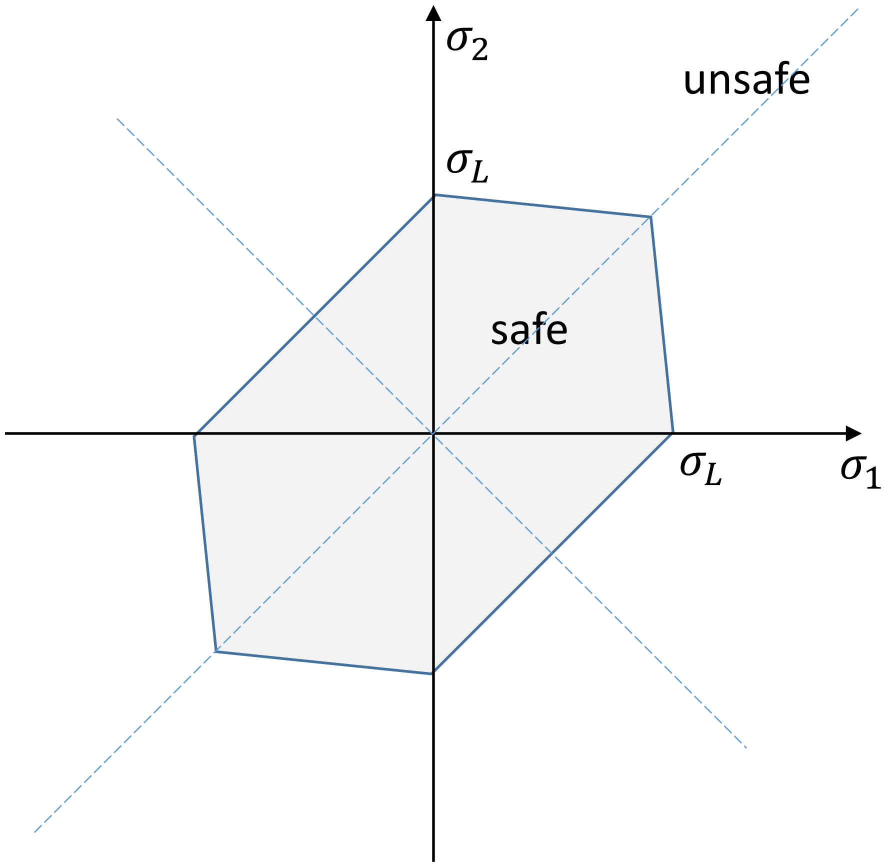
4. For torsion or simple shear:

$$\sigma_{eq} = 2\tau_{xy}$$

At failure:

$$\sigma_{eq} = 2\tau_L = \sigma_L \rightarrow \frac{\sigma_L}{\tau_L} = 2$$

Failure theories: Maximum shear stress (or Tresca criterion)



Failure theories: Maximum distortion energy (or Von Mises criterion)

1. Assumption: the limit state is predicted to occur at the material point when the distortion energy reaches the limit value E_L

2. For the generic multiaxial state of stress the critical condition becomes:

$$E = \frac{1}{12G}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right] = E_L$$

For the uniaxial case:

$$E_{eq} = \frac{1}{12G}\left(\sigma_{eq}\right)^2 \rightarrow E_L = \frac{1}{12G}(\sigma_L)^2$$

Therefore, the limit state (failure) is predicted to occur when:

$$\sigma_{eq} \geq \sigma_L$$

3. In the case of plane stress:

$$\sigma_{eq} = \sqrt{\sigma_x^2+\sigma_y^2-\sigma_x\sigma_y+3\tau_{xy}^2}$$

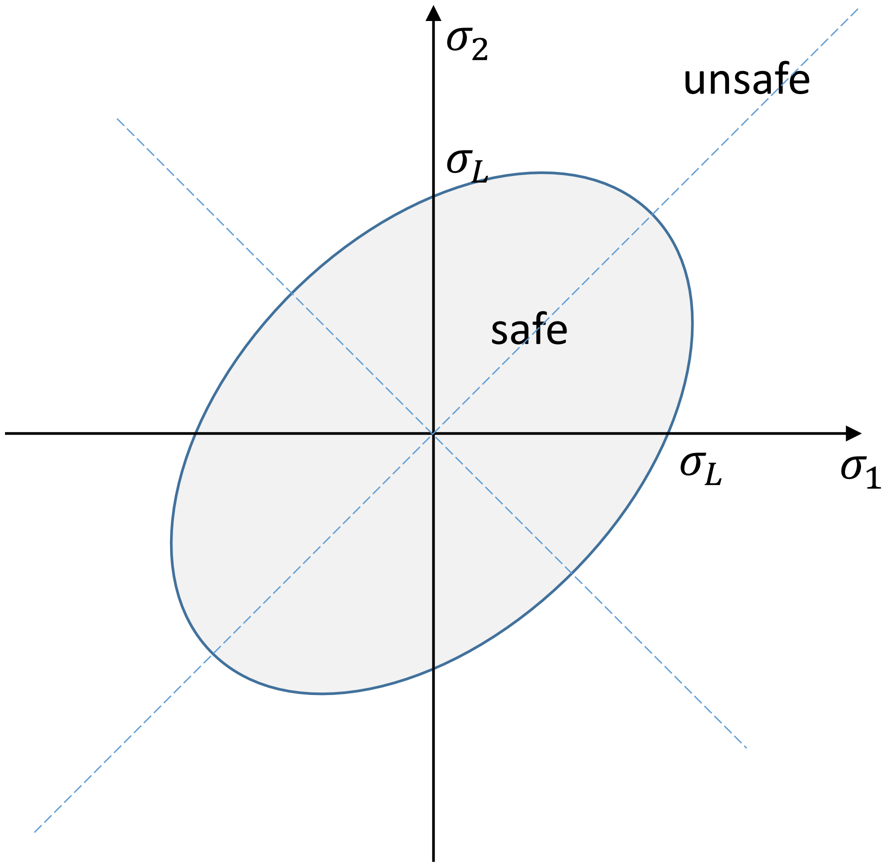
4. For torsion or simple shear:

$$\sigma_{eq} = \sqrt{3}\tau_{xy}$$

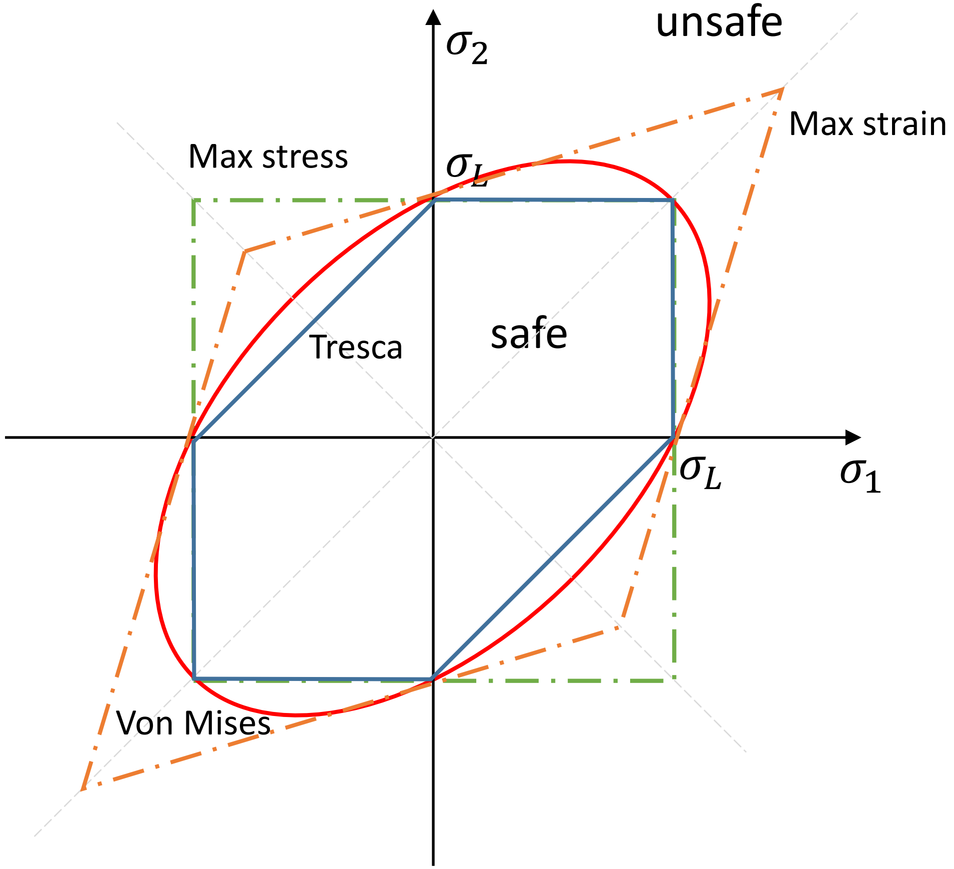
At failure:

$$\sigma_{eq} = \sqrt{3}\tau_L = \sigma_L \rightarrow \frac{\sigma_L}{\tau_L} = \sqrt{3}$$

Failure theories: Maximum shear stress (or Tresca criterion)



Failure theories: Westergaard rapresentation



Failure theories application to materials

DUCTILE	TRESCA VON MISES
BRITTLE	Max stress Mohr

Failure theories application to materials

- Failure theories do not address any specific mechanism of failure
- They are so called “abrupt criteria”
- Do not take into account of the progressive deterioration of the material (damage)
- Are simple but uncoupled with the dissipative processed (i.e. plastic deformation)
- Good for simple, conservative design

Suggested reading

- Brnic, Josip. *Analysis of Engineering Structures and Material Behavior*. John Wiley & Sons, 2018.